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ABSTRACT

The research ~~described in this report~~ deals with the application of abstract techniques in the study of the behavior and (asymptotic) stability of solutions to differential equations of reaction-diffusion type. These equations include models in a cellular control process with positive feedback and a model in gas-liquid reactions. Most of the results ~~as~~ are based on techniques that involve the determination of invariant sets for solutions and the preservation of inequalities between solutions.

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RESEARCH ON NONLINEAR DIFFERENTIAL EQUATIONS

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The main topic of research investigated during this project is the development and application of abstract techniques in order to study the behavior of solutions to nonlinear systems of parabolic equations. The literature generated by this project includes the published articles [1]* - [4], the accepted articles [5] - [6] and the article [7] that is to be submitted. The underlying themes in each of these results are the concepts of invariant sets, inequalities, and stability for solutions to differential equations. Most of the abstract techniques connected with these ideas were developed earlier in the articles [8] - [10]. Therefore, we restrict our attention in this report to the description of results obtained by applying these abstract techniques to particular equations.

In the mathematical modeling of a cellular control process with positive feedback one obtains a system of ordinary differential equations of the following type:

$$
(1)\qquad
\begin{aligned}
z'_1 &= -\alpha_1 z_1 + h_\sigma(z_m), &\quad z_1(0) &= \eta_1 \geq 0\\
z'_2 &= -\alpha_2 z_2 + z_1 & z_2(0) &= \eta_2 \geq 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots & &\\
z'_m &= -\alpha_m z_m + z_{m-1} & z_m(0) &= \eta_m \geq 0
\end{aligned}
$$

---

*Numbers in brackets refer to the bibliography located at the end of this report.

where $\alpha_i > 0$ for $i = 1,\ldots,m$, $\sigma \geq 1$, and

(2) $h_\sigma(r) = r^\sigma(1+r^\sigma)^{-1}$ for all $r \geq 0$.

Several authors have investigated the existence of and the stability properties of nonnegative critical points (equilibrium solutions) to this feedback system. However, the earlier works on this model did not use the fact that solutions to this equation preserve inequalities: if $z(.;\eta)$ is the solution to (1) for each given nonnegative initial value $\eta = (\eta_i)_1^m$, then $z(t;\eta) \geq z(t;\pi)$ (componentwise) for all $t \geq 0$ whenver $\eta \geq \pi \geq \theta$ (componentwise). This observation allows one to both improve and simplify previous results. Also, abstract techniques allow much more general equations to be studied. First we consider the associated reaction-diffusion system.

$$
(3)\quad
\begin{aligned}
&\partial_t u_1 = d_1 \partial_{xx} u_1 - \alpha_1 u_1 + h_\sigma(u_m) \\
&\partial_t u_2 = d_2 \partial_{xx} u_2 - \alpha_2 u_2 + u_1 \qquad\qquad (t,x) \in [0,\infty)x(0,1) \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\partial_t u_m = d_m \partial_{xx} u_m - \alpha_m u_m + u_{m-1}
\end{aligned}
$$

subject to the initial conditions

(4) $u_1(0,x) = \chi_1(x) \geq 0,\ldots,u_m(0,x) = \chi_m(x) \geq 0$ for $x \varepsilon (0,1)$

and the boundary conditions

$$
(5)\quad
\begin{aligned}
&p_o \partial_x u_i(t,0) - q_o u_i(t,0) = p_1 \partial_x u_i(t,1) + q_1 u_i(t,1) = 0 \\
&\text{for all } t > 0 \text{ and } i = 1,\ldots,m
\end{aligned}
$$

where q_o, $q_1 \geq 0$ and $p_1, p_1 \in \{0, 1\}$ are such that $q_o = 1$ whenever $p_o = 0$ and $q_1 = 1$ whenever $p_1 = 0$. Consider also the linear eigenvalue problem

$$
(6)\quad
\begin{aligned}
&\phi''(x) + \lambda\phi(x) = 0, \qquad 0<x<1 \\
&p_o\phi'(0) - q_o\phi(0) = p_1\phi'(1) + q_1\phi(1) = 0
\end{aligned}
$$

and let $\lambda = \lambda_1$ denote the first eigenvalue (note that $\lambda_1 \geq 0$ and that $\lambda_1 = 0$ only if the boundary conditions are $\phi'(0) = \phi'(1) = 0$). The techniques and results in [3] show that the following behavior patterns for solutions to (3) - (5) in valid:

<u>THEOREM 1</u>. Suppose that $\sigma = 1$ in the definition (2) of h_σ, that $d_1,\ldots,d_m$ are positive constants in equation (3), and that λ_1 is the first eigenvalue of equation (6).

(i) If $\prod_{i=1}^{m} (\alpha_i + \lambda_1 d_i) \geq 1$ then for each nonnegative initial value $\chi = (\chi_i)_1^m$ the solution $u = (u_i)_1^m$ to (3) - (5) exists on $[0,\infty) \times [0,1]$ and $u(t,x) \to \theta$ as $t \to \infty$ uniformly for $x \in [0,1]$.

(ii) If $\prod_{i=1}^{m} (\alpha_i + \lambda_1 d_i) < 1$ then there is a unique nontrivial, nonnegative equilibrium solution $\psi = (\psi_i)_1^m$ to (3) - (5):

$$0 = d_1\psi_1''(x) - \alpha_1\psi_1(x) + h_1(\psi_m(x))$$

$$0 = d_2\psi_2''(x) - \alpha_2\psi_2(x) + \psi_1(x) \qquad 0<x<1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 = d_m\psi_m''(x) - \alpha_m\psi_m(x) + \psi_{m-1}(x)$$

and

$$p_0\psi'_i(0) - q_0\psi_i(0) = p_1\psi'_i(1) + q_1\psi_i(1) = 0 \qquad i=1,\ldots,m$$

Moreover, $\psi_i(x) > 0$ for all x and i, and for each nonnegative, nontrivial initial value $\chi = (\chi_i)_1^m$ the solution $u = (u_i)_1^m$ exists on $[0,\infty) \times [0,1]$ and $u(t,x) \to \psi(x)$ as $t \to \infty$, uniformly for $x \in [0,1]$.

It is of interest to note that when $\sigma = 1$ the behavior of the solutions to the ordinary differential equation (1) is precisely the same as that of the reaction-diffusion model if one replaces α_i by $\alpha_i + \lambda_1 d_i$ in (1).

A result similar to Theorem 1 can be obtained by considering time delays in the feedback terms. Again suppose $\sigma = 1$ and consider the delay differential equation.

$$
(7)\quad
\begin{aligned}
&w_1'(t) = -\alpha_1 w_1(t) + h_1(w_m(t-r_m)), && w_1(s) = \phi_1(s), \quad -r_1 \le s \le 0\\
&w_2'(t) = -\alpha_2 w_2(t) + w_1(t-r_1), && w_2(s) = \phi_2(s), \quad -r_2 \le s \le 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&w_m'(t) = -\alpha_m w_m(t) + w_{m-1}(t-r_{m-1}), && w_m(s) = \phi_m(s), \quad -r_m \le s \le 0
\end{aligned}
$$

where $r_i \geq 0$ and ϕ_i is a given nonnegative function on $[-r_i, 0]$ for each $i = 1,\ldots,m$. The results for the system (7) are completely analogous to the corresponding reaction-diffusion system (3) - (5):

<u>THEOREM 2</u>: Consider the time delay differential equation (7).

(i) If $\prod_{i=1}^{m} \alpha_i \geq 1$ then for each nonnegative initial function $\phi = (\phi_i)_1^m$ the solution $w = (w_i)_1^m$ to (7) exist on $[0,\infty)$ and $w(t) \to \theta$ as $t \to \infty$.

(ii) If $\prod_{i=1}^{m} \alpha_i < 1$ then there is a unique nonnegative, nontrivial vector $\delta^* = (\delta_i^*)_1^m$ such that $w(t) \equiv \delta^*$ for $t \geq 0$ (and $\phi_i(s) \equiv \delta_i^*$ on $[-r_i, 0]$ for $i = 1,\ldots,m$) is a constant solution to (7). Moreover, $\delta_i^* > 0$ for all $i = 1,\ldots,m$ and if w is a nontrivial solution to (7) then $w(t) \to \delta^*$ as $t \to \infty$. This result is from the article [6].

Results have also been obtained on the behavior of solution to a nonlinear parabolic system that arises in the study of gas-liquid reactions. The fundamental equation has the form

$$(8)\quad \begin{aligned} \partial_t u(t,x) &= d_1 \partial_{xx} u(t,x) - k\,\gamma(u(t,x),\ v(t,x)) && t>0 \\ \partial_t v(t,x) &= d_2 \partial_{xx} v(t,x) + k\,\gamma(u(t,x),\ v(t,x)) && 0<x<\sigma \end{aligned}$$

subject to the boundary conditions

$$(9)\quad \begin{aligned} u(t,0) &= a_o & \partial_x v(t,0) &= 0 & & t>0 \\ u(t,\sigma) &= 0 & v(t,\sigma) &= 0 \end{aligned}$$

and the initial conditions

$$(10)\quad u(0,x) = u_o(x),\quad v(0,x) = v_o(x), \qquad 0<x<\sigma.$$

Here it is assumed that a_o, b_o and k are given positive constants, σ is either a positive number or $+\infty$, and γ is a real value C^2 function on $[0, a_o] \times [0, b_o]$ such that

(γ1) $\gamma(0,v) = \gamma(u,b_o) = 0$ for all $(u,v) \in [0, a_o] \times [0, b_o]$ and

(γ2) $\partial_u \gamma(u,v) > 0$ and $\partial_v \gamma(u,v) < 0$ for all $(u,v) \in (0, a_o) \times (0, b_o)$.

(The typical example of such a function γ is $\gamma(u,v) = u^p(b_o - v)^q$ where $p,q \geq 1$). The initial conditions u_o and v_o are also assumed to satisfy $0 \leq u_o(x) \leq a_o$ and $0 \leq v_o(x) \leq b_o$ for almost all $x \in [0,\sigma]$.

The analysis of this system fits nicely into an abstract setting. Let $L^1 = L^1[(0,\sigma);\ \mathbb{R}^2]$ be the Banach space of all measurable functions $\phi = (\phi_1,\ \phi_2)$ from $(0,\ \sigma)$ into $\mathbb{R}^2$ such that

$$||\phi||_1 \equiv \int_0^\sigma [|\phi_1(x)| + |\phi_2(x)|]\, dx < \infty$$

and define $D \subset L^1$ by $D = \{(\phi_1,\ \phi_2) \in L^1:\ 0 \leq \phi_1(x) \leq a_o$ and $0 \leq \phi_2(x) \leq b_o$ a.e. on $(0,\ \sigma)\}$. Note that D is a closed convex subset of L^1 (with empty interior!) and that D is bounded only in case $\sigma < \infty$. If $(\phi_1,\ \phi_2)$ and (ψ_1,ψ_2) are in L^1 we write $\phi \leq \psi$ only in case $\phi_1(x) \leq \psi_1(x)$ and $\phi_2(x) \leq \psi_2(x)$ a.e. on $(0,\sigma)$. Concerning the fundamental existence and behavior of solutions to (8) - (10) we have the following result:

THEOREM 3. Suppose that ($\gamma 1$) and ($\gamma 2$) are satisfied. Then for each initial value $(u_o, v_o) \in D$ the system (8) - (10) has a unique solution (u,v) and this solution exists on $(0, \infty) \times (0,\sigma)$ and satisfies $(u(t,.), v(t,.)) \in D$ for all $t \geq 0$ (i.e., $0 \leq u(t,x) \leq a_o$ and $0 \leq v(t,x) \leq b_o$ for all $t \geq 0$ and $0<x<\sigma$). Moreover, if $S(t)(u_o,v_o) \equiv (u(t,.),v(t,.))$ for each initial value (u_o,v_o) in D, then $S = \{S(t):t \geq 0\}$ is a semigroup of nonlinear operators mapping D into D that is non-expansive and order preserving:

(i) $S(t+s)(u_o,v_o) = S(t)S(s)(u_o,v_o)$ for all $t;s \geq 0$ and $(u_o,v_o) \in D$;

(ii) $||S(t)(u_o,v_o) - S(t)(u_1,v_1)||_1 \leq ||(u_o,v_o) - (u_1,v_1)||_1$ for all $t \geq 0$ and (u_o,v_o), $(u_1,v_1) \in D$; and

(iii) $S(t)(u_o,v_o) \leq S(t)(u_1,v_1)$ whenever $t \geq 0$ and (u_o,v_o), $(u_1,v_1) \in D$ with $(u_o,v_o) \leq (u_1,v_1)$.

The proof of this theorem follows from the results in [8] and is indicated in the article [5].

The behavior of solutions to (8) - (10) as $t \to \infty$ has also be investigated in [5] as well as in [1]. The analysis of this behavior divides naturally into two parts: The film theory (when $\sigma < \infty$) and the penetration theory (when $\sigma = \infty$).

THEOREM 4. Suppose that the conditions and notations in Theorem 3 are satisfied.

(i) If $\sigma < \infty$ there is a unique equilibrium solution $\psi = (\psi_1, \psi_2)$ to (8) - (10) [i.e., $d_1\psi_1'' - k\,\gamma(\psi) = 0$ and $d_2\psi_2'' + k\,\gamma(\psi) = 0$ on $[0,\sigma]$, $\psi_1(0) = a_o$, $\psi_2'(0) = 0$, and $\psi_1(\sigma) = \psi_2(\sigma) = 0$] and for each $(u_o,v_o) \in D$, $S(t)(u_o,v_o) \to \psi$ as $t \to \infty$ uniformly for $x \in [0,\sigma]$.

(ii) If $\sigma = \infty$ there is no equilibrium solution to (8) - (10) and for each $(u_o,v_o) \in D$, $[S(t)(u_o,v_o)]\,(x) \to (a_o,b_o)$ as $t \to \infty$ uniformly for x in each bounded subset of $[0, \infty)$.

The results in Theorem 4 as well as more general results can be found in the paper [5].

In [2] abstract techniques are applied to a mathematical model of a gas exchange system. This system can be put in the following form. Suppose that ρ, ℓ, P, σ_1, σ_2, σ_3, and k_1, k_2, k_3, are positive numbers; that $\beta = (\beta_i)_1^3$ and $\gamma = (\gamma_i)_1^3$ are C^1 functions from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^3$; and that $\mu = (\mu_i)_1^3$ and $\nu = (\nu_i)_1^3$ are C^1 on $[0,\infty)$. If $z = (z_i)_1^3$ and $w = (w_i)_1^3$ map $[0,\ell]$ into $\mathbb{R}^3$, the problem is to determine the existence of functions $u = (u_i)_1^3$ and $v = (v_i)_1^3$ from $[0, \infty] \times [0, \ell]$ into $\mathbb{R}^3$ and a function c from $[0,\infty) \times [0,\ell]$ into $\mathbb{R}$ such that the parabolic system

$$(11)\quad \begin{aligned} \partial_t u_i &= \rho \partial_{xx} u_i - \partial_x (c u_i) + \beta_i(u,v) \\ \partial_t v_t &= \sigma_i \partial_{xx} v_i - k_i \partial_x v_i + \gamma_i(u,v) \end{aligned} \qquad \begin{aligned} &t > 0,\ 0<x<\ell \\ &\text{and } i = 1, 2, 3. \end{aligned}$$

is satisfied along with the boundary conditions

$$(12)\quad \begin{aligned} &u(t,0) = \mu(t),\ v(t,0) = \nu(t) \\ &\partial_x u(t,\ell) = \partial_x v(t,x) = c(t,\ell) = 0 \end{aligned} \qquad t>0.$$

the initial conditions

$$(13)\quad u(0,x) = z(x) \text{ and } v(0,x) = w(x) \qquad 0<x<\ell,$$

as well as the <u>side</u> <u>condition</u>

$$(14)\quad u_1(t,x) + u_2(t,x) + u_3(t,x) \equiv P \quad \text{for} \quad t \geq 0,\ 0<x<\ell.$$

The following existence result can be found in [2]:

<u>THEOREM 5</u>. In addition to the suppostions and notations in the preceding paragraph, let R_1, R_2, and R_3 be positive numbers, let

$$\Lambda_1 = \{\xi \in \mathbb{R}^3 : \xi_1 + \xi_2 + \xi_3 = P \text{ and } \xi_1, \xi_2, \xi_3 \geq 0\}$$

$$\Lambda_2 = \{\zeta \in \mathbb{R}^3 : 0 \leq \zeta_i \leq R_i \text{ for } i = 1, 2, 3\},$$

and assume also that the following is valid:

(H1) $(z(x), w(x)) \in \Lambda_1 x \Lambda_2$ for $0<x<\ell$ and $(\mu(t), \nu(t)) \in \Lambda_1 x \Lambda_2$ for all $t \geq 0$; and

(H2) If $(\xi, \eta) \in \Lambda_1 x \Lambda_2$ and $j \in \{1, 2, 3\}$ then $\xi_j = 0$ implies $\beta_j(\xi, \eta) \geq 0$; $\eta_j = 0$ implies $\gamma_j(\xi, \eta) \geq 0$; and $\eta_j = R_j$ implies $\gamma_j(\xi, \eta) \leq 0$.

Then the system (10) - (14) has solution on $[0,\infty) x [0,\ell]$.

As opposed to considering this system as a differential equation with side condition, it is shown in [2] that one can establish the existence of solutions by using the abstract theory of invariant sets for evolution systems (see [9]).

In [4] the (strict, componentwise) positiveness of reaction diffusion systems of the form

(15) $\partial_t u_i = d_i \Delta u_i + F_i(u, \vec{\nabla} u_i)$ on $(0,\infty) x \Omega$ for $i = 1,\ldots,m$

with initial and boundary conditions

(16) $u = \theta$ on $(0,\infty) x \partial\Omega$ and $u = \chi$ on $\{0\} x \Omega$

are considered (as well as extension of such systems) where Ω is a smooth bounded domain in $\mathbb{R}^n$, Δ is the Laplacian operator on Ω, $\vec{\nabla}$ is the gradient operator, and d_i is a positive constant for each $i = 1,\ldots,m$. Criteria is given to insure that if the initial value χ is nontrivial componentwise nonnegative on Ω then each component of the solution u is strictly positive for positive time: $u(o,x) \geq \theta$ (but not identically zero) implies that $u_i(t,x) > 0$ for all $t>0$, $x \in \Omega$ and $i = 1,\ldots m$. This results are obtain by comparison with the ordinary differential equation.

(17) $z'(t) = g(t,z(t))$, $t > 0$

where $g = (g_i)^m_1$ is defined on $[0,\infty) x \mathbb{R}^m$ by

$g_i(t,\xi) = F_i(t,x_o,\xi, \theta)$ for all $(t,\xi) \in [0,\infty) x \mathbb{R}^m$ and $i = 1,\ldots,m$.

(the point x_o is some given point in Ω). This type of maximum principle extends a similar result established in [3].

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